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Low frequency quasi-normal modes of AdS black holes*

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Abstract: We calculate analytically low frequency quasi-normal modes of gravitational perturbations of AdS Schwarzschild black holes in d dimensions. We arrive at analytic expressions which are in agreement with their counterparts from linearized hydrodynamics in $S^{d-2} \times \mathbb{R}$, in accordance with the AdS/CFT correspondence. Our results are also in good agreement with results of numerical calculations.

Keywords: Black Holes, AdS-CFT Correspondence, Gauge-gravity correspondence, Strong Coupling Expansion.

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Contents

1. Introduction

Quasi-normal modes (QNMs) determine the late-time evolution of black hole perturbations. They have been extensively studied in asymptotically Anti-de Sitter (AdS) space-times [1] in hopes of shedding some light on the Anti-de Sitter - conformal field theory (AdS/CFT) correspondence. Analytic expressions for asymptotic frequencies were derived in [2, 3] (adapting the monodromy argument proposed in [4] and extended to first order in [5]) for arbitrary dimension. First-order corrections to these analytic expressions were obtained in [6] in good agreement with numerical results [7].

At the other end of the spectrum, the low frequency modes can be used to probe the behavior of the gauge theory on the boundary of AdS in the hydrodynamic limit [8]. This is particularly interesting in connection with heavy ion collisions at RHIC (see, e.g., [9] and references therein) and the LHC. It appears that the quark-gluon plasma (QGP) that forms is strongly coupled raising the possibility that it may possess a gravity dual.

The gauge fluid analyzed in [8] corresponds to a dual black hole of flat horizon in five dimensions which is obtained in the near-horizon limit of three-branes. More generally, in AdS_d, one obtains black hole solutions of metric

$$ds^2 = -f_k(r)dt^2 + \frac{dr^2}{f_k(r)} + r^2 d\Sigma^2_{k,d-2} \ , \quad f_k(r) = \frac{r^2}{R^2} + k - \frac{2\mu}{r^{d-3}} \tag{1.1}$$

where $k = 0, \pm 1$. For $k = 0$, $d\Sigma^2_{k,d-2}$ represents the flat metric of $\mathbb{R}^{d-2}$ and the boundary ($r \to \infty$) is the flat Minkowski space $R^{d-2,1}$. For $k = +1$, $d\Sigma^2_{k,d-2} = d\Omega^2_{d-2}$ in S^{d-2} and the boundary is $S^{d-2} \times \mathbb{R}$ (Einstein Universe). For $k = -1$, the boundary is $\mathbb{R} \times H^{d-2}$, where H^{d-2} is a hyperbolic space (static open Universe) [10]. The former case ($k = 0$) is easily seen to be the large black hole limit ($\mu \to \infty$) of the other two ($k = \pm 1$).

In [11], the QGP was analyzed in terms of a "conformal soliton flow" obtained from a black hole with $k = +1$ by mapping the boundary of AdS_5, $S^3 \times \mathbb{R}$, to the four-dimensional flat Minkowski space by a conformal transformation. The QNMs were calculated numerically and led to an elliptic flow coefficient and thermalization time that compared well with experimental results. This provided a physical motivation for the study of $k = +1$ black holes in AdS_5 and the attendant analysis of hydrodynamics on an Einstein Universe ($S^3 \times \mathbb{R}$).

Here we calculate the low frequency QNMs of AdS Schwarzschild black holes in d dimensions analytically extending the discussion of ref. [8]. We find agreement with numerical results in four [7] and five [11] dimensions. Our analytic expressions also agree with their counterparts obtained from hydrodynamics on the Einstein Universe $S^{d-2} \times \mathbb{R}$ [11, 12] in accordance with the AdS/CFT correspondence. We discuss gravitational perturbations using the Master Equation of Ishibashi and Kodama [13], including vector (section 2), scalar (section 3) and tensor (section 4) modes. Our conclusions are summarized in section 5.

2. Vector perturbations

In this section, we discuss vector perturbations and the attendant QNMs in the AdS Schwarzschild black hole background with $k = +1$ in d dimensions. The metric is obtained from eq. (1.1),

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2 \ , \quad f(r) = \frac{r^2}{R^2} + 1 - \frac{2\mu}{r^{d-3}} \ . \tag{2.1}$$

Scalar and tensor perturbations will be discussed in the subsequent sections. Perturbations are described by linearized Einstein equations and different types have been shown to decouple by Ishibashi and Kodama [13]. They arrived at Master Equations for the respective perturbations which will be our starting point.

To simplify the notation, we shall set the AdS radius $R = 1$. The radius of the horizon is related to the mass parameter by

$$2\mu = r_H^{d-1}\left(1 + \frac{1}{r_H^2}\right) \tag{2.2}$$

Also, the Hawking temperature is given by

$$T_H = \frac{f'(r_H)}{4\pi} = \frac{(d-1)r_H^2 + d - 3}{4\pi r_H} \tag{2.3}$$

The radial wave equation for gravitational perturbations in the black-hole background (2.1) can be cast into a Schrödinger-like form,

$$-\frac{d^2\Psi}{dr_*^2} + V[r(r_*)]\Psi = \omega^2\Psi \ , \tag{2.4}$$

in terms of the tortoise coordinate defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)} \ . \tag{2.5}$$

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It is also useful to introduce the distance

$$\bar{r}_* = r_*(\infty) - r_*(0) = \int_0^\infty \frac{dr}{f(r)} \tag{2.6}$$

A short calculation yields [3]

$$\bar{r}_* = \frac{\pi}{(d-1)r_H}\left(\cot\frac{\pi}{d-1} + i\right) + O(1/r_H^2) \tag{2.7}$$

The potential V is determined by the type of perturbation and may be deduced from the Master Equation of Ishibashi and Kodama [13]. For vector perturbations, we obtain [3]

$$V(r) = V_{\mathsf{V}}(r) \equiv f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} - \frac{rf'''(r)}{2(d-3)}\right\} \tag{2.8}$$

Evidently, the potential vanishes at the horizon ($V(r_H) = 0$, since $f(r_H) = 0$). This is the case for all types of perturbation.

The asymptotic form of QNMs was found in [2, 3] analytically. Subsequently, it was shown in [6] how the approximation may be improved upon by a perturbative expansion. One obtains

$$\begin{aligned} \omega_n &= \omega_n^{(0)} + \omega_n^{(1)} + \dots \\ \omega_n^{(0)}\bar{r}_* &= \left(n + \frac{d-3}{4}\right)\pi + \frac{1}{2i}\ln 2 \\ \omega_n^{(1)}\bar{r}_* &= \mathcal{A}_{\mathsf{V}}\frac{e^{-\frac{i\pi}{2(d-2)}}\cos^4\frac{\pi}{2(d-2)}}{2\pi^2 r_H^2}\frac{(\omega_n^{(0)}/r_H)^{-\frac{d-3}{d-2}}}{[2(d-2)(1+1/r_H^2)]^{\frac{1}{d-2}}}\frac{\Gamma(\frac{1}{d-2})\Gamma^4(\frac{d-3}{2(d-2)})(d-3)}{(d-1)} \end{aligned} \tag{2.9}$$

where

$$\mathcal{A}_{\mathsf{V}} = \frac{\ell(\ell+d-3)}{d-2} + \frac{d^2-8d+13}{2(2d-15)} \tag{2.10}$$

Thus, the first-order correction is $O(n^{-\frac{d-3}{d-2}})$. It should also be noted that the first-order correction is suppressed by a factor of $1/r_H^2$. So for large black holes, the zeroth-order contribution provides a good approximation to all modes, not just the high overtones. Moreover, the zeroth-order term is independent of both the angular momentum quantum number ℓ and the type of perturbation. This leads to a mild dependence on ℓ and the type of perturbation of all QNMs for large black holes ($r_H \gtrsim 1$). Finally, since $\bar{r}_* \sim 1/r_H$, the frequencies are proportional to the radius of the horizon.

In particular, for $d = 5$, the QNMs are given by

$$\frac{\omega_n}{r_H} = \left(2n + 1 + i\frac{\ln 2}{\pi}\right)(1-i) + O(1/r_H^2) \tag{2.11}$$

Numerically, we find

$$\frac{\omega_1}{r_H} \approx \left(3 + i\frac{\ln 2}{\pi}\right)(1-i) = 3.221 - 2.779i\ , \quad \frac{\Delta\omega_n}{r_H} \equiv \frac{\omega_{n+1}-\omega_n}{r_H} \approx 2(1-i) \tag{2.12}$$

These results are independent of the type of perturbation and the quantum number ℓ. They agree well with numerical results [11].

Even though the above expressions include high as well as low frequencies, they do not include the lowest overtones. The latter may be obtained using the method of [8] and they correspond to the hydrodynamic behavior of the gauge theory fluid.

To obtain analytic expressions for the lowest overtones, it is convenient to introduce the coordinate

$$u = \left(\frac{r_H}{r}\right)^{d-3} \tag{2.13}$$

The wave equation (2.4) becomes

$$-(d-3)^2 u^{\frac{d-4}{d-3}} \hat{f}(u) \left(u^{\frac{d-4}{d-3}} \hat{f}(u)\Psi'\right)' + \hat{V}_{\mathsf{V}}(u)\Psi = \hat{\omega}^2\Psi \ , \qquad \hat{\omega} = \frac{\omega}{r_H} \tag{2.14}$$

where prime denotes differentiation with respect to u and we have defined

$$\begin{aligned} \hat{f}(u) &\equiv \frac{f(r)}{r^2} \\ &= 1 - u^{\frac{2}{d-3}}\left(u - \frac{1-u}{r_H^2}\right) \end{aligned} \tag{2.15}$$

$$\begin{aligned} \hat{V}_{\mathsf{V}}(u) &\equiv \frac{V_{\mathsf{V}}}{r_H^2} \\ &= \hat{f}(u)\left\{\hat{L}^2 + \frac{(d-2)(d-4)}{4}u^{-\frac{2}{d-3}}\hat{f}(u) - \frac{(d-1)(d-2)\left(1+\frac{1}{r_H^2}\right)}{2}u\right\} \end{aligned} \tag{2.16}$$

where

$$\hat{L}^2 = \frac{\ell(\ell+d-3)}{r_H^2} \tag{2.17}$$

Let us first consider the large black hole limit $r_H \to \infty$ keeping $\hat{\omega}$ and $\hat{L}$ fixed (small). Factoring out the behavior at the horizon ($u = 1$)

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} F(u) \tag{2.18}$$

the wave equation simplifies to

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega},\hat{L}}F = 0 \tag{2.19}$$

where

$$\begin{aligned} \mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\ \mathcal{B}_{\hat{\omega}} &= -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\ \mathcal{C}_{\hat{\omega},\hat{L}} &= \hat{L}^2 + \frac{(d-2)[d-4-3(d-2)u^{\frac{d-1}{d-3}}]}{4}u^{-\frac{2}{d-3}} \\ &\quad -\frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2\frac{\hat{\omega}^2}{(d-1)^2}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\ &\quad -(d-3)\frac{i\hat{\omega}}{d-1}\frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2\frac{i\hat{\omega}}{d-1}\frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \end{aligned}$$

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For small $\hat{\omega}$, $\hat{L}$, eq. (2.19) may be solved perturbatively by writing it as

$$(\mathcal{H}_0 + \mathcal{H}_1)F = 0 \tag{2.20}$$

where

$$\begin{aligned} \mathcal{H}_0 F &\equiv \mathcal{A}F'' + \mathcal{B}_0 F' + \mathcal{C}_{0,0} F \\ \mathcal{H}_1 F &\equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F' + (\mathcal{C}_{\hat{\omega},\hat{L}} - \mathcal{C}_{0,0})F \end{aligned} \tag{2.21}$$

Expanding the wavefunction perturbatively,

$$F = F_0 + F_1 + \dots \tag{2.22}$$

at zeroth order we have

$$\mathcal{H}_0 F_0 = 0 \tag{2.23}$$

whose acceptable solution is

$$F_0 = u^{\frac{d-2}{2(d-3)}} \tag{2.24}$$

It is regular at both the horizon ($u = 1$) and the boundary ($u = 0$, or $\Psi \sim r^{-\frac{d-2}{2}} \to 0$ as $r \to \infty$). The Wronskian (up to an arbitrary multiplicative constant) is

$$\mathcal{W} = \frac{1}{u^{\frac{d-4}{d-3}}(1 - u^{\frac{d-1}{d-3}})} \tag{2.25}$$

Another linearly independent solution is

$$\check{F}_0 = F_0 \int \frac{\mathcal{W}}{F_0^2} \tag{2.26}$$

It is unacceptable because it diverges at both the horizon ($\check{F}_0 \sim \ln(1-u)$ for $u \approx 1$) and the boundary ($\check{F}_0 \sim u^{-\frac{d-4}{2(d-3)}}$ for $u \approx 0$, or $\Psi \sim r^{\frac{d-4}{2}} \to \infty$ as $r \to \infty$). It may be expressed in terms of hypergeometric functions but its explicit form is not needed for our purposes (first-order perturbation theory).

At first order we have

$$\mathcal{H}_0 F_1 = -\mathcal{H}_1 F_0 \tag{2.27}$$

whose solution may be written as

$$F_1 = F_0 \int \frac{\mathcal{W}}{F_0^2} \int \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} \tag{2.28}$$

The limits of the inner integral may be adjusted at will because this amounts to adding an arbitrary amount of the unacceptable solution (2.26). To ensure regularity at the horizon, we should choose one of the limits at $u = 1$ (the integrand is regular at the horizon, by design). Then the behavior of the wavefunction (2.28) at the boundary ($u = 0$) is given by

$$F_1 = \check{F}_0 \int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} + \dots \tag{2.29}$$

where we omitted regular terms. The coefficient of the singularity ought to vanish,

$$\int_0^1 \frac{F_0 \mathcal{H}_1 F_0}{\mathcal{A}\mathcal{W}} = 0 \tag{2.30}$$

which imposes a constraint on the parameters (dispersion relation) of the form

$$\mathbf{a}_0 \hat{L}^2 - i\mathbf{a}_1 \hat{\omega} - \mathbf{a}_2 \hat{\omega}^2 = 0 \tag{2.31}$$

After some algebra, we arrive at explicit expressions for the coefficients,

$$\mathbf{a}_0 = \frac{d-3}{d-1} \ , \quad \mathbf{a}_1 = d-3 \tag{2.32}$$

The coefficient $\mathbf{a}_2$ may also be found explicitly for each dimension d, but it cannot be written as a function of d in closed form. However, it does not contribute to the dispersion relation at lowest order. E.g., for $d = 4, 5$, we obtain, respectively

$$\mathbf{a}_2 = \frac{65}{108} - \frac{1}{3}\ln 3 \ , \quad \frac{5}{6} - \frac{1}{2}\ln 2 \tag{2.33}$$

The quadratic in $\hat{\omega}$ equation (2.31) has two solutions,

$$\hat{\omega}_0 \approx -i\frac{\hat{L}^2}{d-1} \ , \quad \hat{\omega}_1 \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\hat{L}^2}{d-1} \tag{2.34}$$

where we omitted terms of higher order in $\hat{L}^2$. In terms of the frequency ω and the quantum number ℓ, they may be written respectively as

$$\omega_0 \approx -i\frac{\ell(\ell+d-3)}{(d-1)r_H} \ , \quad \frac{\omega_1}{r_H} \approx -i\frac{d-3}{\mathbf{a}_2} + i\frac{\ell(\ell+d-3)}{(d-1)r_H^2} \tag{2.35}$$

The smaller of the two, ω_0, is inversely proportional to the radius of the horizon and is not included in the spectrum (2.9) obtained earlier by different means [2, 3, 6]. The other solution, ω_1 is a crude estimate of the first overtone in the spectrum (2.9). Numerically, for $d = 5$, we obtain using (2.33)

$$\frac{\omega_1}{r_H} \approx -4.109i + i\frac{\ell(\ell+2)}{4r_H^2} \tag{2.36}$$

to be compared with the numerical value (2.12).

It should be noted that the crude estimate (2.36) already exhibits two important features: ω_1 is proportional to r_H and the dependence on ℓ is of order $1/r_H^2$, as expected from (2.9). The approximation may be improved by including higher-order terms in the perturbative expansion. Inclusion of higher orders also increases the degree of the polynomial in the dispersion relation (2.31) whose roots then yield approximate values of more QNMs. Thus, this method reproduces the spectrum (2.9) albeit not in an efficient way.

The lowest frequency ω_0 (eq. (2.35)) agrees with the result obtained from a flat black hole (eq. (1.1) with $k = 0$) [8] if we identify

$$\ell(\ell+d-3) \equiv q^2 \ , \quad \vec{q} \in \mathbb{R}^{d-2} \tag{2.37}$$

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where $\vec{q}$ is the momentum. Indeed, (2.35) may then be written as

$$\omega_0 = -i\,\frac{q^2}{4\pi T_H} \tag{2.38}$$

where we also used (2.3) in the large r_H limit. Eq. (2.38) is a diffusion equation with diffusion constant $\mathcal{D} = \frac{1}{4\pi T_H}$.

The above discussion may be extended to black holes of finite size in a straightforward manner by treating finite-size effects as a perturbation (assuming $1/r_H$ is small). Thus at first order, we need to replace $\mathcal{H}_1$ (eq. (2.21)) by

$$\mathcal{H}_1' = \mathcal{H}_1 + \frac{1}{r_H^2}\mathcal{H}_H \tag{2.39}$$

where

$$\mathcal{H}_H F \equiv \mathcal{A}_H F'' + \mathcal{B}_H F' + \mathcal{C}_H F \tag{2.40}$$

The coefficients may be easily deduced by collecting $O(1/r_H^2)$ terms in the exact wave equation given by (2.14), (2.15) and (2.16). We obtain

$$\begin{aligned} \mathcal{A}_H &= -2(d-3)^2 u^2 (1-u) \\ \mathcal{B}_H &= -(d-3)u\left[(d-3)(2-3u) - (d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}}u^{\frac{d-1}{d-3}}\right] \\ \mathcal{C}_H &= \frac{d-2}{2}\left[d-4-(2d-5)u-(d-1)\frac{1-u}{1-u^{\frac{d-1}{d-3}}}u^{\frac{d-1}{d-3}}\right] \end{aligned} \tag{2.41}$$

Interestingly, the zeroth order wavefunction F_0 (eq. (2.24)) is an eigenfunction of $\mathcal{H}_H$,

$$\mathcal{H}_H F_0 = -(d-2)F_0 \tag{2.42}$$

therefore, the first-order finite-size effect is a simple shift of the angular momentum (eq. (2.17))

$$\hat{L}^2 \to \hat{L}^2 - \frac{d-2}{r_H^2} \tag{2.43}$$

Consequently, the QNMs of lowest frequency (2.35) are modified to

$$\omega_0 = -i\frac{\ell(\ell+d-3)-(d-2)}{(d-1)r_H} + O(1/r_H^2) \tag{2.44}$$

For $d = 4, 5$, we have respectively,

$$\omega_0 = -i\frac{(\ell-1)(\ell+2)}{3r_H}\ , \qquad -i\frac{(\ell+1)^2-4}{4r_H} \tag{2.45}$$

in agreement with numerical results ([7] and [11], respectively).

Next we discuss the role the lowest frequency mode plays in the AdS/CFT correspondence. The dual to the AdS Schwarzschild black hole is a gauge theory fluid on the boundary of AdS ($S^{d-2} \times \mathbb{R}$). To find the dual of vector perturbations, one ought to consider the fluid dynamics ansatz

$$u_i = \mathcal{K} e^{-i\Omega\tau} \mathbb{V}_i \tag{2.46}$$

where u_i is the (small) velocity of a point in the fluid and $\mathbb{V}_i$ is a vector harmonic on S^{d-2}. Demanding that this ansatz satisfy the standard equations of linearized hydrodynamics, one arrives at a constraint on the frequency of the perturbation Ω which yields [11, 12]

$$\Omega = -i\frac{\ell(\ell+d-3)-(d-2)}{(d-1)r_H} + O(1/r_H^2) \tag{2.47}$$

in perfect agreement with its dual counterpart (2.44).

Comparing with the discussion in [8], hydrodynamics in the Einstein Universe $S^{d-2} \times \mathbb{R}$ reduces to its counterpart in Minkowski space $\mathbb{R}^{d-2,1}$ in the large ℓ limit. As remarked above, we may identify $\ell(\ell+d-3)$ with q^2 (eq. (2.37)) and then we recover the diffusion equation (2.38). Eq. (2.47) contains a correction which is sub-leading in ℓ, as expected. The correction becomes important for small ℓ, which are the modes with the lowest frequencies. These modes dominate at later times and cannot adequately be described by a flat black hole approximation.

3. Scalar perturbations

Scalar perturbations are also governed by a Schrödinger-like wave equation (2.4) with the potential given by [13]

$$\begin{aligned} V_{\mathsf{S}}(r) = \frac{f(r)}{4r^2}&\left[\ell(\ell+d-3)-(d-2)+\frac{(d-1)(d-2)\mu}{r^{d-3}}\right]^{-2} \\ \times&\left\{\frac{d(d-1)^2(d-2)^3\mu^2}{r^{2d-8}} - \frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{r^{d-5}}\right. \\ &+(d-4)(d-6)[\ell(\ell+d-3)-(d-2)]^2r^2 + \frac{2(d-1)^2(d-2)^4\mu^3}{r^{3d-9}} \\ &+\frac{4(d-1)(d-2)(2d^2-11d+18)[\ell(\ell+d-3)-(d-2)]\mu^2}{r^{2d-6}} \\ &+\frac{(d-1)^2(d-2)^2(d-4)(d-6)\mu^2}{r^{2d-6}} \\ &-\frac{6(d-2)(d-6)[\ell(\ell+d-3)-(d-2)]^2\mu}{r^{d-3}} \\ &-\frac{6(d-1)(d-2)^2(d-4)[\ell(\ell+d-3)-(d-2)]\mu}{r^{d-3}} \\ &\left.+4[\ell(\ell+d-3)-(d-2)]^3 + d(d-2)[\ell(\ell+d-3)-(d-2)]^2\right\} \end{aligned} \tag{3.1}$$

QNMs are asymptotically the same as the QNMs (2.9) of vector perturbations [2, 3]. They differ at first order in the perturbative expansion. For scalar modes we obtain [6]

$$\omega_n^{(1)}\bar{r}_* = \mathcal{A}_{\mathsf{S}} \frac{e^{-\frac{i\pi}{2(d-2)}}\cos^4\frac{\pi}{2(d-2)}}{2\pi^2 r_H^2} \frac{(\omega_n^{(0)}/r_H)^{-\frac{d-3}{d-2}}}{[2(d-2)(1+1/r_H^2)]^{\frac{1}{d-2}}}\Gamma(\frac{1}{d-2})\Gamma^4(\frac{d-3}{2(d-2)}) \tag{3.2}$$

where

$$\mathcal{A}_{\mathsf{S}} = \frac{(d^2-7d+14)[\ell(\ell+d-3)-(d-2)]}{(d-1)(d-2)^2} + \frac{2d^3-24d^2+94d-116}{4(2d-5)(d-2)} \tag{3.3}$$

Again, we see the ℓ-dependence entering at $O(1/r_H^2)$, as for vector perturbations. Also, the above set of frequencies does not exhaust the spectrum as it leaves out the lowest frequency mode. To find it, we work as before. Changing variables to (2.13), we may write the wave equation as in (2.14) with $\hat{V}_{\mathsf{V}}$ replaced by

$$\begin{aligned}\hat{V}_{\mathsf{S}}(u) = &\frac{\hat{f}(u)}{4}\left[\hat{m}+\left(1+\frac{1}{r_H^2}\right)u\right]^{-2} \\ &\times\Bigg\{d(d-2)\left(1+\frac{1}{r_H^2}\right)^2 u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}\left(1+\frac{1}{r_H^2}\right)u^{\frac{d-5}{d-3}} \\ &\quad +(d-4)(d-6)\hat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2\left(1+\frac{1}{r_H^2}\right)^3 u^3 \\ &\quad +2(2d^2-11d+18)\hat{m}\left(1+\frac{1}{r_H^2}\right)^2 u^2 \\ &\quad +\frac{(d-4)(d-6)\left(1+\frac{1}{r_H^2}\right)^2}{r_H^2}u^2 - 3(d-2)(d-6)\hat{m}^2\left(1+\frac{1}{r_H^2}\right)u \\ &\quad -\frac{6(d-2)(d-4)\hat{m}\left(1+\frac{1}{r_H^2}\right)}{r_H^2}u + 2(d-1)(d-2)\hat{m}^3 + d(d-2)\frac{\hat{m}^2}{r_H^2}\Bigg\}\end{aligned} \tag{3.4}$$

where

$$\hat{m} = 2\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H^2} = \frac{2(\ell+d-2)(\ell-1)}{(d-1)(d-2)r_H^2} \tag{3.5}$$

In the large black hole limit $r_H \to \infty$ with $\hat{m}$ fixed, the potential simplifies to

$$\begin{aligned}\hat{V}_{\mathsf{S}}^{(0)}(u) = &\frac{1-u^{\frac{d-1}{d-3}}}{4(\hat{m}+u)^2}\Bigg\{d(d-2)u^{\frac{2d-8}{d-3}} - 6(d-2)(d-4)\hat{m}u^{\frac{d-5}{d-3}} \\ &\quad +(d-4)(d-6)\hat{m}^2 u^{-\frac{2}{d-3}} + (d-2)^2 u^3 \\ &\quad +2(2d^2-11d+18)\hat{m}u^2 - 3(d-2)(d-6)\hat{m}^2 u + 2(d-1)(d-2)\hat{m}^3\Bigg\}\end{aligned} \tag{3.6}$$

In addition to the singularities at the end points ($u = 0, 1$), the scalar wave equation has an additional singularity due to the double pole of the scalar potential at $u = -\hat{m}$. It is

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desirable to factor out the behavior not only at the horizon, but also at the boundary and the pole of the scalar potential. We therefore define

$$\Psi = (1-u)^{-i\frac{\hat{\omega}}{d-1}} \frac{u^{\frac{d-4}{2(d-3)}}}{\hat{m}+u} F(u) \tag{3.7}$$

In terms of $F(u)$, the wave equation for scalar perturbations in the large black hole limit reads

$$\mathcal{A}F'' + \mathcal{B}_{\hat{\omega}}F' + \mathcal{C}_{\hat{\omega}}F = 0 \tag{3.8}$$

where

$$\begin{aligned}
\mathcal{A} &= -(d-3)^2 u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \\
\mathcal{B}_{\hat{\omega}} &= -(d-3)u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \left[\frac{d-4}{u} - \frac{2(d-3)}{\hat{m}+u}\right] \\
&\quad -(d-3)[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}} - 2(d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u} \\
\mathcal{C}_{\hat{\omega}} &= -u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}}) \left[-\frac{(d-2)(d-4)}{4u^2} - \frac{(d-3)(d-4)}{u(\hat{m}+u)} + \frac{2(d-3)^2}{(\hat{m}+u)^2}\right] \\
&\quad - \left[\left\{d-4-(2d-5)u^{\frac{d-1}{d-3}}\right\} u^{\frac{d-5}{d-3}} + 2(d-3)\frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{1-u}\right] \left[\frac{d-4}{2u} - \frac{d-3}{\hat{m}+u}\right] \\
&\quad -(d-3)\frac{i\hat{\omega}}{d-1} \frac{[d-4-(2d-5)u^{\frac{d-1}{d-3}}]u^{\frac{d-5}{d-3}}}{1-u} - (d-3)^2 \frac{i\hat{\omega}}{d-1} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2} \\
&\quad + \frac{\hat{V}_{\mathsf{S}}^{(0)}(u) - \hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}} + (d-3)^2 \frac{\hat{\omega}^2}{(d-1)^2} \frac{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})}{(1-u)^2}
\end{aligned}$$

To calculate the QNMs perturbatively, we define the zeroth-order wave equation as in (2.23) with

$$\mathcal{H}_0 F \equiv \mathcal{A}F'' + \mathcal{B}_0 F' \tag{3.9}$$

The acceptable zeroth-order solution is (arbitrarily normalized)

$$F_0(u) = 1 \tag{3.10}$$

which is plainly regular at all singular points ($u = 0, 1, -\hat{m}$). On account of (3.7), it corresponds to a wavefunction vanishing at the boundary ($\Psi \sim r^{-\frac{d-4}{2}}$ as $r \to \infty$, using (2.13)). The Wronskian is

$$\mathcal{W} = \frac{(\hat{m}+u)^2}{u^{\frac{2d-8}{d-3}}(1-u^{\frac{d-1}{d-3}})} \tag{3.11}$$

The unacceptable solution is

$$\check{F}_0 = \int \mathcal{W} \tag{3.12}$$

and can be written in terms of hypergeometric functions. It has a singularity at the boundary, $\check{F}_0 \sim u^{-\frac{d-5}{d-3}}$ for $u \approx 0$. Using (3.7), we deduce $\Psi \sim r^{\frac{d-6}{2}} \to \infty$ as $r \to \infty$ for $d \geq 6$.

For $d = 5$, both wavefunctions vanish at the boundary. The wavefunction corresponding to F_0 behaves as $r^{-1/2}$ whereas the one corresponding to $\check{F}_0$ behaves as $r^{-1/2}\ln r$. The latter is seen to lead to non-vanishing deformations of the boundary metric [11] and is therefore unacceptable.

For $d = 4$ the roles of F_0 and $\check{F}_0$ are reversed. The wavefunction corresponding to F_0 approaches a constant at the boundary whereas the one corresponding to $\check{F}_0$ behaves as $1/r \to 0$. The former reads

$$\Psi_0 = \frac{1}{\hat{m} + u} \tag{3.13}$$

where we used (3.7) and (3.10). At the boundary ($u = 0$), it obeys the Robin boundary condition

$$\frac{d\Psi_0}{du} + \frac{1}{\hat{m}}\Psi_0 = 0 \tag{3.14}$$

In [12], it was shown that this is the correct boundary condition for vanishing deformation of the boundary metric for $d = 4$ instead of the customary Dirichlet boundary condition $\Psi = 0$ at $u = 0$. Adopting this boundary condition for $d = 4$ amounts to building perturbation theory around F_0 and discarding $\check{F}_0$, as in all other dimensions.

At first order in perturbation theory, working as in the case of vector modes, we arrive at the constraint

$$\int_0^1 \frac{\mathcal{C}_{\hat{\omega}}}{\mathcal{A}\mathcal{W}} = 0 \tag{3.15}$$

where we used (2.30), (3.10) and

$$\mathcal{H}_1 F_0 \equiv (\mathcal{B}_{\hat{\omega}} - \mathcal{B}_0)F_0' + \mathcal{C}_{\hat{\omega}} F_0 = \mathcal{C}_{\hat{\omega}} \tag{3.16}$$

The first-order constraint (3.15) may be written as a dispersion relation

$$\mathbf{a}_0 - \mathbf{a}_1 i\hat{\omega} - \mathbf{a}_2\hat{\omega}^2 = 0 \tag{3.17}$$

After some algebra, we obtain

$$\mathbf{a}_0 = \frac{d-1}{2}\,\frac{1+(d-2)\hat{m}}{(1+\hat{m})^2} \quad , \quad \mathbf{a}_1 = \frac{d-3}{(1+\hat{m})^2} \quad , \quad \mathbf{a}_2 = \frac{1}{\hat{m}}\left\{1 + O(\hat{m})\right\} \tag{3.18}$$

For small $\hat{m}$, the quadratic equation (3.17) yields the solutions

$$\hat{\omega}_0^{\pm} \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\,\hat{m}} \tag{3.19}$$

The two solutions are related to each other by $\hat{\omega}_0^{+} = -\hat{\omega}_0^{-*}$, which is a general symmetry of the spectrum. Neither solution is included in the spectrum obtained from asymptotic QNMs. To obtain approximations to those modes, we need to consider higher orders in perturbation theory. Notice also that unlike vector modes, these scalar lowest frequency modes have finite real part. Using (3.5), we may express the frequencies in terms of the quantum number ℓ as

$$\omega_0^{\pm} \approx -i(d-3)\,\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H} \pm \sqrt{\frac{\ell(\ell+d-3)-(d-2)}{d-2}} \tag{3.20}$$

Thus the real part is independent of r_H whereas the imaginary part is inversely proportional to r_H (for $r_H \gtrsim 1$).

This result agrees with the dispersion relation obtained through a flat black hole [8]. Indeed, with the identification (2.37), eq. (3.20) may be written as

$$\omega_0^\pm = -i\frac{d-3}{d-2}\,\frac{q^2}{4\pi T_H} \pm v_s q \ , \qquad v_s = \frac{1}{\sqrt{d-2}} \tag{3.21}$$

showing that the speed of sound v_s has the correct value for a conformal fluid.

Finite size effects may be added perturbatively. At first order, they amount to a shift of the coefficient $\mathbf{a}_0$ in the dispersion relation (3.17),

$$\mathbf{a}_0 \to \mathbf{a}_0 + \frac{1}{r_H^2}\mathbf{a}_H \tag{3.22}$$

Working as in the vector case, after some tedious but straightforward algebra, we obtain

$$\mathbf{a}_H = \frac{1}{\hat{m}}\left\{1 + O(\hat{m})\right\} \tag{3.23}$$

The modified dispersion relation yields the modes

$$\hat{\omega}_0^\pm \approx -i\frac{d-3}{2}\,\hat{m} \pm \sqrt{\frac{d-1}{2}\,\hat{m}+1} \tag{3.24}$$

correcting (3.19). Explicitly,

$$\omega_0^\pm \approx -i(d-3)\,\frac{\ell(\ell+d-3)-(d-2)}{(d-1)(d-2)r_H} \pm \sqrt{\frac{\ell(\ell+d-3)}{d-2}} \tag{3.25}$$

correcting (3.20). This is in agreement with numerical results [11, 12].

Turning to the AdS/CFT correspondence, we ought to perturb the gauge theory fluid on the boundary of AdS ($S^{d-2}\times\mathbb{R}$) using the ansatz

$$u_i = \mathcal{K}e^{-i\Omega\tau}\nabla_i\mathbb{S} \ , \qquad \delta p = \mathcal{K}'e^{-i\Omega\tau}\mathbb{S} \tag{3.26}$$

where u_i is the (small) velocity of a point in the fluid, δp is the pressure perturbation and $\mathbb{S}_i$ is a scalar harmonic on S^{d-2}. Demanding that this ansatz satisfy the standard equations of linearized hydrodynamics, one arrives at an expression for the frequency of the perturbation Ω [11, 12] which is in perfect agreement with our analytic result (3.25).

Comparing with the flat black hole case [8], eq. (3.25) includes a correction to (3.21) which is subleading in ℓ, as in the vector case (see remarks at end of section 2). These corections are important for small ℓ modes which dominate the late time behavior of the perturbation.

4. Tensor perturbations

Finally we discuss tensor perturbations. In this case the potential is given by [13]

$$V_{\mathsf{T}}(r) = f(r)\left\{\frac{\ell(\ell+d-3)}{r^2} + \frac{(d-2)(d-4)f(r)}{4r^2} + \frac{(d-2)f'(r)}{2r}\right\} \tag{4.1}$$

It leads to the same asymptotic form of QNMs as in the other two cases (vector and scalar) [2, 3]. The first-order correction to the asymptotic expression is [6]

$$\omega_n^{(1)}\bar{r}_* = \mathcal{A}_{\mathsf{T}} \frac{e^{-\frac{i\pi}{2(d-2)}}\cos^4\frac{\pi}{2(d-2)}}{2\pi^2 r_H^2} \frac{(\omega_n^{(0)}/r_H)^{-\frac{d-3}{d-2}}}{[2(d-2)(1+1/r_H^2)]^{\frac{1}{d-2}}} \Gamma\left(\frac{1}{d-2}\right)\Gamma^4\left(\frac{d-3}{2(d-2)}\right) \quad (4.2)$$

where

$$\mathcal{A}_{\mathsf{T}} = \frac{\ell(\ell+d-3)}{d-2} + \frac{(d-3)^2}{2(2d-5)} \quad (4.3)$$

to be compared with the vector (2.9) and scalar (3.2) cases. Once again we obtain a mild $O(1/r_H^2)$ dependence on the quantum number ℓ.

Unlike the other two cases, this constitutes the entire spectrum. To see this, let us change variables to (2.13) and go to the large black hole limit. The wave equation simplifies to

$$-(d-3)^2(u^{\frac{2d-8}{d-3}} - u^3)\Psi'' - (d-3)[(d-4)u^{\frac{d-5}{d-3}} - (2d-5)u^2]\Psi' + \left\{\hat{L}^2 + \frac{d(d-2)}{4}u^{-\frac{2}{d-3}} + \frac{(d-2)^2}{4}u - \frac{\hat{\omega}^2}{1-u^{\frac{d-1}{d-3}}}\right\}\Psi = 0 \quad (4.4)$$

The zeroth order equation is obtained from (4.4) by setting $\hat{L}$ and $\hat{\omega}$ to zero. The resulting equation may be solved exactly. Two linearly independent solutions are ($\Psi = F_0$ at zeroth order)

$$F_0(u) = u^{\frac{d-2}{2(d-3)}} \quad , \quad \check{F}_0(u) = u^{-\frac{d-2}{2(d-3)}} \ln\left(1 - u^{\frac{d-1}{d-3}}\right) \quad (4.5)$$

Neither behaves nicely at both ends ($u = 0, 1$), therefore both are unacceptable. It is not possible to build a perturbation theory to calculate small frequencies.

This negative result is in agreement with numerical results and is also in accordance with the AdS/CFT correspondence [11]. Indeed, there is no ansatz that can be built from tensor spherical harmonics $\mathbb{T}_{ij}$ (similar to the vector (2.46) and scalar (3.26) cases) satisfying the linearized hydrodynamic equations because of the conservation and tracelessness properties of $\mathbb{T}_{ij}$ [11].

5. Conclusion

We calculated analytically low frequency QNMs of gravitational perturbations of AdS Schwarzschild black holes in arbitrary dimension using the Master Equation of Ishibashi and Kodama [13]. We noted that low frequency modes are well approximated by asymptotic expressions for large black holes (with radius of horizon $r_H \gtrsim 1$ in units such that the AdS radius $R = 1$) [2, 3]. These expressions ($\omega_n^{(0)}$ in eq. (2.9)) are independent of the type of perturbation and the angular momentum quantum number ℓ. The dependence on ℓ enters at first-order perturbation theory and is $O(1/r_H^2)$ [6]. The first-order contribution, $\omega_n^{(1)}$, for vector, scalar and tensor perturbations is given by eqs. (2.9), (3.2) and (4.2), respectively.

Asymptotic expressions do not in general yield the entire spectrum. To find the lowest frequency mode, we applied the method in ref. [8]. We also included the effects of a finite size black hole. We arrived at explicit analytic expressions in the case of vector (eq. (2.44)) and scalar (eq. (3.25)) modes. For tensor modes, this method does not yield a new mode (the asymptotic series exhausts the spectrum). Our analytic expressions were in agreement with numerical results [7, 11, 12]. They also agreed perfectly with the results from linearized hydrodynamics of the gauge theory fluid on the boundary of AdS space [11, 12] in accordance with the AdS/CFT correspondence.

It would be interesting to extend these calculations to a less symmetric configuration that would better fit the experimental setup of heavy ion collisions at RHIC and the LHC. Then a comparison with experimental results would enhance our understanding of gauge theory fluid dynamics at strong coupling.

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References

[1] J.S.F. Chan and R.B. Mann, *Scalar wave falloff in asymptotically anti-de Sitter backgrounds*, *Phys. Rev.* **D 55** (1997) 7546 [gr-qc/9612026];
G.T. Horowitz and V.E. Hubeny, *Quasinormal modes of AdS black holes and the approach to thermal equilibrium*, *Phys. Rev.* **D 62** (2000) 024027 [hep-th/9909056];
G.T. Horowitz, *Comments on black holes in string theory*, *Class. and Quant. Grav.* **17** (2000) 1107 [hep-th/9910082];
B. Wang, C.-Y. Lin and E. Abdalla, *Quasinormal modes of reissner-nordstroem anti-de Sitter black holes*, *Phys. Lett.* **B 481** (2000) 79 [hep-th/0003295];
B. Wang, C. Molina and E. Abdalla, *Evolving of a massless scalar field in reissner-nordstroem anti-de Sitter spacetimes*, *Phys. Rev.* **D 63** (2001) 084001 [hep-th/0005143];
V. Cardoso and J.P.S. Lemos, *Scalar, electromagnetic and weyl perturbations of btz black holes: quasi normal modes*, *Phys. Rev.* **D 63** (2001) 124015 [gr-qc/0101052];
V. Cardoso and J.P.S. Lemos, *Quasi-normal modes of Schwarzschild Anti-de Sitter black holes: electromagnetic and gravitational perturbations*, *Phys. Rev.* **D 64** (2001) 084017 [gr-qc/0105103];
V. Cardoso and J.P.S. Lemos, *Quasi-normal modes of toroidal, cylindrical and planar black holes in Anti-de Sitter spacetimes*, *Class. and Quant. Grav.* **18** (2001) 5257 [gr-qc/0107098];
G. Siopsis, *Large mass expansion of quasi-normal modes in AdS_5*, *Phys. Lett.* **B 590** (2004) 105 [hep-th/0402083];
G. Siopsis, *On quasi-normal modes and the AdS_5/CFT_4 correspondence*, *Nucl. Phys.* **B 715** (2005) 483 [hep-th/0407157];
M. Giammatteo and J.-l. Jing, *Dirac quasinormal frequencies in Schwarzschild-AdS space-time*, *Phys. Rev.* **D 71** (2005) 024007 [gr-qc/0403030];
B. Wang, C.-Y. Lin and C. Molina, *Quasinormal behavior of massless scalar field perturbation in reissner-nordstroem Anti-de Sitter spacetimes*, *Phys. Rev.* **D 70** (2004) 064025 [hep-th/0407024];
P.K. Kovtun and A.O. Starinets, *Quasinormal modes and holography*, *Phys. Rev.* **D 72** (2005) 086009 [hep-th/0506184];
R.A. Konoplya, *Massive vector field perturbations in the Schwarzschild background: stability and unusual quasinormal spectrum*, *Phys. Rev.* **D 73** (2006) 024009 [gr-qc/0509026].

[2] V. Cardoso, J. Natario and R. Schiappa, *Asymptotic quasinormal frequencies for black holes in non-asymptotically flat spacetimes*, *J. Math. Phys.* **45** (2004) 4698 [hep-th/0403132].

[3] J. Natario and R. Schiappa, *On the classification of asymptotic quasinormal frequencies for D-dimensional black holes and quantum gravity*, *Adv. Theor. Math. Phys.* **8** (2004) 1001 [hep-th/0411267].

[4] L. Motl and A. Neitzke, *Asymptotic black hole quasinormal frequencies*, *Adv. Theor. Math. Phys.* **7** (2003) 307 [hep-th/0301173].

[5] S. Musiri and G. Siopsis, *Perturbative calculation of quasi-normal modes of Schwarzschild black holes*, *Class. and Quant. Grav.* **20** (2003) L285 [hep-th/0308168].

[6] S. Musiri, S. Ness and G. Siopsis, *Perturbative calculation of quasi-normal modes of AdS Schwarzschild black holes*, *Phys. Rev.* **D 73** (2006) 064001 [hep-th/0511113].

[7] V. Cardoso, R. Konoplya and J.P.S. Lemos, *Quasinormal frequencies of Schwarzschild black holes in Anti-de Sitter spacetimes: a complete study on the asymptotic behavior*, *Phys. Rev.* **D 68** (2003) 044024 [gr-qc/0305037].

[8] D.T. Son and A.O. Starinets, *Minkowski-space correlators in AdS/CFT correspondence: recipe and applications*, *JHEP* **09** (2002) 042 [hep-th/0205051];
G. Policastro, D.T. Son and A.O. Starinets, *From AdS/CFT correspondence to hydrodynamics*, *JHEP* **09** (2002) 043 [hep-th/0205052].

[9] B. Muller and J.L. Nagle, *Results from the relativistic heavy ion collider*, *Ann. Rev. Nucl. Part. Sci.* **56** (2006) 93 [nucl-th/0602029].

[10] D. Birmingham, *Topological black holes in Anti-de Sitter space*, *Class. and Quant. Grav.* **16** (1999) 1197 [hep-th/9808032].

[11] J.J. Friess, S.S. Gubser, G. Michalogiorgakis and S.S. Pufu, *Expanding plasmas and quasinormal modes of Anti-de Sitter black holes*, *JHEP* **04** (2007) 080 [hep-th/0611005].

[12] G. Michalogiorgakis and S.S. Pufu, *Low-lying gravitational modes in the scalar sector of the global AdS_4 black hole*, *JHEP* **02** (2007) 023 [hep-th/0612065].

[13] H. Kodama and A. Ishibashi, *A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions*, *Prog. Theor. Phys.* **110** (2003) 701 [hep-th/0305147].

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